Cosmological Issues in $F(T)$ Gravity Theory

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We review recent developments on cosmology in extended teleparallel gravity, called "$F(T)$ gravity" with T the torsion scalar in teleparallelism. We explore various cosmological aspects of $F(T)$ gravity including the evolution of the equation of state for the universe, finite-time future singularities, thermodynamics, and four-dimensional effective $F(T)$ gravity theories coming from the higher-dimensional Kaluza-Klein (KK) and Randall-Sundrum (RS) theories.

I. INTRODUCTION

The fact that the current accelerated the cosmic expansion is currently accelerating has been supported by various recent cosmological observations including Type Ia Supernovae (SNe Ia), cosmic microwave background (CMB) radiation, baryon acoustic oscillations (BAO), large scale structure (LSS), and weak lensing effects (see recent results acquired from the Planck satellite [1, 2] as well as the Wilkinson Microwave anisotropy probe (WMAP) [3, 4]). There exist the following two main procedures to account for the late-time cosmic acceleration: The introducttion of "dark energy" and the extension of gravity, e.g., the so-called $F(R)$ gravity (for recent reviews on dark energy and modified gravity, see, for example, [5] and [6–15], respectively).

As a formulation for gravity, there has been proposed "teleparallelism" where the gravity theory is described by using the Weitzenböck connection (for a recent detailed review, see [16]). This has been considered to be an alternative gravitational theory to general relativity. This gravity theory is written with the torsion scalar T, and not the scalar curvature R defined with the Levi-Civita connection [17–19] as in general relativity. Recently, it has been found that as in $F(R)$ gravity, not only inflation in the early universe [20, 21] but also the late-time cosmic acceleration [22–25] can occur in the so-called $F(T)$ gravity, which is an extended version of the original teleparallelism.

In this paper, we review main cosmological consequences in $F(T)$ gravity obtained in Refs. [24–27]. First, we investigate the evolution of the equation of state (EoS) for dark energy [24, 25]. We construct an $F(T)$ gravity model in which the crossing of the phantom divide can happen[1]. This phenomenon has been suggested with cosmological observations in Refs. [36–40]. Second, we demonstrate that the finite-time future singularities [41–43] can appear in $F(T)$ gravity [26]. In addition, $F(T)$ gravity models with realizing the finite-time future singularities are reconstructed. We find that the finite-time future singularities can be cured by adding a power-law term T^β with $\beta > 1$, for instance, a T^2 term. The same approach has been used for Loop quantum cosmology [44]. Furthermore, we examine $F(T)$ models in which inflation, the ΛCDM model, Little Rip scenario [45–54], and Pseudo-Rip scenario [55] can be realized. Third, we derive four-dimensional effective $F(T)$ gravity theories from the five-dimensional Kaluza-Klein (KK) [56–58] and Randall-Sundrum (RS) [59, 60] models [27]. It is also demonstrated that inflation and the late-time cosmic acceleration can occur in the former four-dimensional effective $F(T)$ gravity theory and the latter RS model, respectively. We use units of $k_{\mathrm{B}} = c = \hbar = 1$ and denote the gravitational constant $8\pi G$ by $\kappa^2 \equiv 8\pi/M_{\mathrm{Pl}}^2$ with the Planck mass of $M_{\mathrm{Pl}} = G^{-1/2} = 1.2 \times 10^{19}$GeV.

The paper is organized as follows. In Sec. II, we consider the cosmological evolution of the EoS for dark energy. In Sec. III, we analyze the finite-time future singularities and reconstruct $F(T)$ gravity models where these finite-time future singularities can appear. We also examine $F(T)$ models to realize various cosmological scenarios. In Sec. IV, we deduce four-dimensional effective $F(T)$ gravity theories from both the KK theories and the RS models. In Sec. V, summary is presented.

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[1] In Refs. [28–35], such an $F(R)$ gravity model with the crossing of the phantom divide has been reconstructed.

TABLE I: Conditions that there exist the finite-time future singularities for H in Eqs. (III.1) and (III.2), those for ρ_{DE} and P_{DE}, and the evolutions of H and $\dot{H}$ for $t \to t_{\mathrm{s}}$.

$q(\neq 0, -1)$ [Type of singularities]	$H\ (t \to t_{\mathrm{s}})$	$\dot{H}\ (t \to t_{\mathrm{s}})$	ρ_{DE}	P_{DE}
$q \geq 1$ [Type I ("Big Rip")]	$H \to \infty$	$\dot{H} \to \infty$	$J_1 \neq 0$	$J_1 \neq 0$ or $J_2 \neq 0$
$0 < q < 1$ [Type III]	$H \to \infty$	$\dot{H} \to \infty$	$J_1 \neq 0$	$J_1 \neq 0$
$-1 < q < 0$ [Type II ("Sudden")]	$H \to H_{\mathrm{s}}$	$\dot{H} \to \infty$		$J_2 \neq 0$
$q < -1$ (q is non-integer) [Type IV]	$H \to H_{\mathrm{s}}$	$\dot{H} \to 0$		
		Divergence of higher derivatives of H		

II.   COSMOLOGICAL EVOLUTIONS

In this section, we explain $F(T)$ gravity and examine the cosmological evolutioin of the EoS for dark energy based on the main results in Ref. [24]. The purpose is to construct an $F(T)$ gravity model in which the crossing of the phantom divide can occur as suggested by resent cosmologcal observations.

A.   Teleparallelism

We first explain the formulation of teleparallelism. The metric is described as $g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu$. Here, η_{AB} is the metric in the Minkowski space-time, $e_A(x^\mu)$ are orthonormal tetrad components ($A = 0, 1, 2, 3$) at points x^μ of the manifold in the tangent space, μ, $\nu = 0, 1, 2, 3$ show coordinate indices on the manifold, and e^μ_A corresponds to the tangent vector of the manifold. The Lagrangian is written with the torsion scalar T. This is different from the case for general relativity, in which the Lagrangian is expressed by using the scalar curvature R. The torsion scalar T is defined as $T \equiv S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu}$, where $T^\rho{}_{\mu\nu} \equiv e^\rho_A \left(\partial_\mu e^A_\nu - \partial_\nu e^A_\mu\right)$ is the torsion tensor and $S_\rho{}^{\mu\nu} \equiv (1/2)\left(K^{\mu\nu}{}_\rho + \delta^\mu_\rho T^{\alpha\nu}{}_\alpha - \delta^\nu_\rho T^{\alpha\mu}{}_\alpha\right)$ with $K^{\mu\nu}{}_\rho \equiv -(1/2)\left(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}\right)$ the contorsion tensor.

The Lagrangian of pure teleparallelism is written by the torsion scalar T. This has been extended to an appropriate function of T to realize inflation and the late-time cosmic acceleration. This concept is the same as $F(R)$ gravity, where the Einstein-Hilbert action written by the scalar curvature R is promoted to an appropriate function of R. Accordingly, the action of $F(T)$ gravity is represented as [23] $I = \int d^4 x |e| \left[F(T)/\left(2\kappa^2\right) + \mathcal{L}_{\mathrm{M}}\right]$ with $|e| = \det\left(e^A_\mu\right) = \sqrt{-g}$ and $\mathcal{L}_{\mathrm{M}}$ the matter Lagrangian. If $F(T) = T$, this action is equivalent to that for pure teleparallelism.

We assume the flat Friedmann-Lemaître-Robertson-Walker (FLRW) space-time. The metric is given by $ds^2 = dt^2 - a^2(t) \sum_{i=1,2,3} \left(dx^i\right)^2$. Here, $a(t)$ is the scale factor, and the Hubble parameter reads $H = \dot{a}/a$, where the dot means the time derivative. In the FLRW background, we obtain the expressions of the metric $g_{\mu\nu} = \mathrm{diag}(1, -a^2, -a^2, -a^2)$, the tetrad components $e^A_\mu = (1, a, a, a)$, and the torsion scalar $T = -6H^2$.

Moreover, in this background, the gravitational field equations are written as $H^2 = \left(\kappa^2/3\right)\left(\rho_{\mathrm{M}} + \rho_{\mathrm{DE}}\right)$ and $\dot{H} = -\left(\kappa^2/2\right)\left(\rho_{\mathrm{M}} + P_{\mathrm{M}} + \rho_{\mathrm{DE}} + P_{\mathrm{DE}}\right)$. Here, ρ_{M} and P_{M} are the energy density and pressure for all of the matters, i.e., the perfect fluids, respectively. The continuity equation for the perfect fluid becomes $\dot{\rho}_{\mathrm{M}} + 3H\left(\rho_{\mathrm{M}} + P_{\mathrm{M}}\right) = 0$. Furthermore, ρ_{DE} and P_{DE} are the energy density and pressure for the dark energy components, respectively, given by $\rho_{\mathrm{DE}} = \left[1/\left(2\kappa^2\right)\right] J_1$ and $P_{\mathrm{DE}} = -\left[1/\left(2\kappa^2\right)\right]\left(4J_2 + J_1\right)$ with $J_1 \equiv -T - F(T) + 2TF'(T)$ and $J_2 \equiv \left(1 - F'(T) - 2TF''(T)\right)\dot{H}$, where the prime denotes the derivative with respect to T as $F'(T) \equiv dF(T)/dT$ and $F''(T) \equiv d^2F(T)/dT^2$. The continuity equation for the dark energy components reads $\dot{\rho}_{\mathrm{DE}} + 3H\left(\rho_{\mathrm{DE}} + P_{\mathrm{DE}}\right) = 0$.

B.   Crossing of the phantom divide

As an $F(T)$ gravity model in which the crossing of the phantom divide can occur, we obtain

$$F(T) = T + \gamma\left\{T_0\left(\frac{uT_0}{T}\right)^{-1/2}\ln\left(\frac{uT_0}{T}\right) - T\left[1 - \exp\left(\frac{uT_0}{T}\right)\right]\right\}, \tag{II.1}$$

$$\gamma \equiv \frac{1 - \Omega_{\mathrm{m}}^{(0)}}{2u^{-1/2} + \left[1 - (1 - 2u)\exp(u)\right]}, \tag{II.2}$$

with T_0 the present value of the torsion scalar T and u a constant. In addition, $\Omega_{\mathrm{m}}^{(0)} \equiv \rho_{\mathrm{m}}^{(0)}/\rho_{\mathrm{crit}}^{(0)}$. Here, $\rho_{\mathrm{m}}^{(0)}$ is the current energy density of non-relativistic matter, and $\rho_{\mathrm{crit}}^{(0)} = 3H_0^2/\kappa^2$ is the current critical density, where H_0 is the Hubble parameter at the present time. The model in Eq. (II.1) consists of both the logarithmic and exponential terms. The EoS for dark energy is defined as $w_{\mathrm{DE}} \equiv P_{\mathrm{DE}}/\rho_{\mathrm{DE}}$. It is seen that for the model in Eq. (II.1), the EoS for dark energy w_{DE} evolves from $w_{\mathrm{DE}} > -1$ to $w_{\mathrm{DE}} < -1$, and thus the crossing of the phantom divide line $w_{\mathrm{DE}} = -1$ can happen. We remark that this manner is opposite to the representative behavior for $F(R)$ gravity models.

Furthermore, it can numerically be demonstrated that for the model in Eq. (II.1), first the density parameter of radiation $\Omega_{\mathrm{r}} \equiv \rho_{\mathrm{r}}/\rho_{\mathrm{crit}}$ dominates, and then the density parameter of non-relativistic matter $\Omega_{\mathrm{m}} \equiv \rho_{\mathrm{m}}/\rho_{\mathrm{crit}}$ is dominant, and eventually the density parameter of dark energy $\Omega_{\mathrm{DE}} \equiv \rho_{\mathrm{DE}}/\rho_{\mathrm{crit}}$ becomes much larger than the density parameters of radiation and non-relativistic matter around the present time. Here, ρ_{r}, ρ_{m}, ρ_{DE}, and $\rho_{\mathrm{crit}} \equiv 3H^2/\kappa^2$ are the energy density of radiation, that of non-relativistic matter, that of dark energy, and the critical density, respectively. Hence, in the model in Eq. (II.1), the dark energy dominated stage, which follows the radiation dominated stage and the matter dominated stage, can be realized. Moreover, through the statistical analysis with the recent cosmological observational data in terms of SNe Ia, BAO, and the CMB radiation, we derive the observational constraints on the model parameters of the model in Eq. (II.1). As a result, we find that the model in Eq. (II.1) can fit the observational data well. In Ref. [61], other $F(T)$ gravity models in which the crossing of the phantom divide can occur have been built up.

III. FINITE-TIME FUTURE SINGULARITIES

In this section, we show that the finite-time future singularities can occur in $F(T)$ gravity by reviewing the consequences in Ref. [26], We also reconstruct $F(T)$ gravity models in which the finite-time future singularities appear.

A. Four types of the finite-time future singularities

For the FLRW space-time, the effective EoS is given by [6, 7] $w_{\mathrm{eff}} \equiv P_{\mathrm{eff}}/\rho_{\mathrm{eff}} = -1 - 2\dot{H}/\left(3H^2\right)$, where $\rho_{\mathrm{eff}} \equiv 3H^2/\kappa^2$ and $P_{\mathrm{eff}} \equiv -\left(2\dot{H} + 3H^2\right)/\kappa^2$ are the energy density and pressure of all of the energy components in the universe, respectively. When the dark energy density becomes dominant over the energy density of non-relativistic matters, the following approximation is satisfied: $w_{\mathrm{DE}} \approx w_{\mathrm{eff}}$. In what follows, we explore such a situation in order to examine the cosmic evolution when there appear the finite-time future singularities at $t = t_{\mathrm{s}}$. If $\dot{H} < 0 \ (> 0)$, the universe is in the non-phantom [i.e., quintessence] (phantom) phase with $w_{\mathrm{eff}} > -1 \ (< -1)$. For $w_{\mathrm{eff}} = -1$, we have $\dot{H} = 0$, namely, the cosmological constant.

The finite-time future singularities are classified into four types [41]. Type I: When $t \to t_{\mathrm{s}}$, $a \to \infty$, $\rho_{\mathrm{eff}} \to \infty$ and $|P_{\mathrm{eff}}| \to \infty$. This types includes the case that ρ_{eff} and P_{eff} become finite values at $t = t_{\mathrm{s}}$ [62]. Type II: When $t \to t_{\mathrm{s}}$, $a \to a_{\mathrm{s}}$, $\rho_{\mathrm{eff}} \to \rho_{\mathrm{s}}$ and $|P_{\mathrm{eff}}| \to \infty$, where $a_{\mathrm{s}}(\neq 0)$ and ρ_{s} are constants. Type III: When $t \to t_{\mathrm{s}}$, $a \to a_{\mathrm{s}}$, $\rho_{\mathrm{eff}} \to \infty$ and $|P_{\mathrm{eff}}| \to \infty$. Type IV: When $t \to t_{\mathrm{s}}$, $a \to a_{\mathrm{s}}$, $\rho_{\mathrm{eff}} \to 0$, $|P_{\mathrm{eff}}| \to 0$. Here, the higher derivatives of H also diverge, and the case that ρ_{eff} and/or $|P_{\mathrm{eff}}|$ approach finite values in the limit $t \to t_{\mathrm{s}}$ is included. Type I and Type II are known as "Big Rip" [63, 64] and "Sudden" [65, 66] singularities.

B. Conditions for the finite-time future singularities to appear

We suppose that H is described as [67]

$$H \sim \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}} - t)^q} \quad \text{for} \ q > 0 \,, \tag{III.1}$$

$$H \sim H_{\mathrm{s}} + \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}} - t)^q} \quad \text{for} \ q < -1, \ -1 < q < 0 \,, \tag{III.2}$$

with $h_{\mathrm{s}}(> 0)$, $H_{\mathrm{s}}(> 0)$, and $q(\neq 0, -1)$ constants. Since the value of H has to be a real number, we examine the range $0 < t < t_{\mathrm{s}}$. In Table I, we summarize the conditions that there exist the finite-time future singularities for H in Eqs. (III.1) and (III.2), those for ρ_{DE} and P_{DE}, and the evolutions of H and $\dot{H}$ for $t \to t_{\mathrm{s}}$.

If H is represented as in Eqs. (III.1) and (III.2), we reconstruct $F(T)$ gravity models, in which the finite-time future singularities happen, by using the procedure [68, 69]. It is seen that in the flat FLRW universe, both of two gravitational filed equations can be met when $F(T)$ is given by the following power-law expression: $F(T) = AT^\alpha$,

TABLE II: Conditions that the parameters in a power-law expression for $F(T)$, for which the finite-time future singularities exist, and the forms of a power-low the correction term $F_{\rm c}(T) = BT^\beta$ curing the finite-time future singularities.

$q(\neq 0, -1)$ [Type of singularities]	Consequence	$F(T) = AT^\alpha$ ($A \neq 0$, $\alpha \neq 0$)	$F_{\rm c}(T) = BT^\beta$ ($B \neq 0$, $\beta \neq 0$)
$q \geq 1$ [Type I ("Big Rip")]	appears	$\alpha < 0$	$\beta > 1$
$0 < q < 1$ [Type III]	—	$\alpha < 0$	$\beta > 1$
$-1 < q < 0$ [Type II ("Sudden")]	—	$\alpha = 1/2$	$\beta \neq 1/2$
$q < -1$ (q is non-integer) [Type IV]	appears	$\alpha = 1/2$	$\beta \neq 1/2$

TABLE III: H and $F(T)$ for which (i) inflation in the early universe, (ii) the ΛCDM model, (iii) Little Rip cosmology and (iv) Pseudo-Rip cosmology can be realized. Here, $h_{\rm inf}$, Λ, $H_{\rm LR}$, ζ, and $H_{\rm PR}$ are constants.

Scenario	H	$F(T)$
(i) Power-law inflation (when $t \to 0$)	$H = h_{\rm inf}/t$, $h_{\rm inf}(>1)$	$F(T) = AT^\alpha$, $\alpha < 0$ or $\alpha = 1/2$
(ii) ΛCDM model or exponential inflation	$H = \sqrt{\Lambda/3} = {\rm constant}$, $\Lambda > 0$	$F(T) = T - 2\Lambda$, $\Lambda > 0$
(iii) Little Rip scenario (when $t \to \infty$)	$H = H_{\rm LR} \exp(\zeta t)$, $H_{\rm LR} > 0$ and $\zeta > 0$	$F(T) = AT^\alpha$, $\alpha < 0$ or $\alpha = 1/2$
(iv) Pseudo-Rip scenario	$H = H_{\rm PR} \tanh(t/t_0)$, $H_{\rm PR} > 0$	$F(T) = A\sqrt{T}$

where $A(\neq 0)$ and $\alpha(\neq 0)$ are constants. Furthermore, we find a correction term $F_{\rm c}(T)$ curing the finite-time future singularities, given by $F_{\rm c}(T) = BT^\beta$ with $B(\neq 0)$ and $\beta(\neq 0)$ constants. It is known that the finite-time future singularities can be removed by the quadratic term (namely, $\beta = 2$) [6, 7] for $F(R)$ gravity and non-local gravity [70]. As a result, for $F(T) = AT^\alpha + BT^\beta$, which is the summation of the original and correction terms, two gravitational filed equations cannot simultaneously be satisfied. It follows from this fact that the finite-time future singularities can be removed by such a power-law correction term. We show the conditions that the parameters in a power-law expression for $F(T)$, for which the finite-time future singularities exist, and the forms of a power-low the correction term curing the finite-time future singularities in Table II. From this table, it is clearly seen that Type I and IV singularities can appear.

C. Various cosmological scenarios

Moreover, we examine which kinds of the finite-time future singularities occur in each cosmology. If the absolute value of q is large enough, the finite-time future singularities can appear. We also explore $F(T)$ gravity models in which the following cosmological scenarios can be realized: (i) Power-law inflation, (ii) The ΛCDM model, (iii) Little Rip scenario [45–47], and (iv) Pseudo-Rip scenario [55]. The expressions of H and $F(T)$ in the scenarios shown above are presented in Table III.

In addition, we consider Little Rip scenario, which is a kind of a mild phantom cosmology. The motivation of this scenario is to remove the finite-time future singularities including a Big Rip singularity. In this scenario, the dark energy density grows as the universe evolves, whereas the EoS for dark energy $w_{\rm DE}$ becomes close to $w_{\rm DE} = -1$ from $w_{\rm DE} < -1$. The special feature of this scenario is that at some future time, bound structures are dissolved because an inertial force operating objects becomes large. Such a phenomenon is the so-called Little Rip.

As another related cosmology, we study Pseudo-Rip scenario. With the Hubble parameter, cosmological scenarios can be classified [55]. (i) Power-law inflation: $H(t) \to \infty$ for $t \to 0$. (ii) The ΛCDM model (or Exponential inflation): $H(t) = H(t_0) = {\rm constant}$, where t_0 is the current time. (iii) Little Rip scenario: $H(t) \to \infty$ for $t \to \infty$. (iv) Pseudo-Rip scenario (a phantom cosmology approaching de Sitter expansion asymptotically: $H(t) \to H_\infty < \infty$ for $t \to \infty$ with $t \geq t_0$ and $H_\infty(> 0)$ a constant. For a Big Rip singularity, $H(t) \to \infty$, $t \to t_{\rm s}$, as depicted in Table I.

We note that the EoS parameter $w_{\rm DE}$ for dark energy, the deceleration parameter $q_{\rm dec} \equiv -\ddot{a}/(aH^2)$, the jerk parameter $j \equiv \dddot{a}/(aH^3)$ and the snark parameter $s \equiv (j-1)/[3(q_{\rm dec} - 1/2)]$ [71, 72] are used to observationally constrain the dark energy models. For the ΛCDM model, we have $(w_{\rm DE}, q_{\rm dec}, j, s) = (-1, -1, 1, 0)$. In the flat universe, there have been proposed $w_{\rm DE} = -1.10 \pm 0.14\,(68\%\,{\rm CL})$ [3]. By using these parameters, especially, the observational constraints on the models parameters can be derived. For example, with the observational value of $w_{\rm DE}$, the constraints on $H_{\rm LR}$ and $H_{\rm PR}$ shown in Table III can be derived. In Little Rip scenario, we obtain $H_{\rm LR} \geq [2H_0/(3e)](1/0.24) = 1.50 \times 10^{-42}\,{\rm GeV}$, where $H_0 = 2.1h \times 10^{-42}\,{\rm GeV}$ [73] with $h = 0.7$ [3, 74] is the current Hubble parameter and $e = 2.71828$, and $\chi \equiv H_0/(H_{\rm LR}e) \leq 0.36$. On the other hand, for Pseudo Rip scenario, we get

$H_{\mathrm{PR}} \geq (2H_0/3)\left[4/\left(e - e^{-1}\right)^2\right](1/0.24) = 2.96 \times 10^{-42}\,\mathrm{GeV}$ and $\delta \equiv H_0/H_{\mathrm{PR}} \leq 0.497196$. Hence, Little Rip scenario with $\chi \ll 1$ and Pseudo Rip scenario with an appropriate value of δ can be compatible with the ΛCDM model.

D. Inertial force

In the expanding universe, the relative acceleration between two points separated by a distance l is given by $l\ddot{a}/a$, where a is the scale factor. Suppose that there exists a particle with mass m at each of the points, an observer at one of the masses would measure an inertial force on the other mass. We assume that there are two particles (A) and (B) with its mass m and the distance between them is l. The inertial force F_{inert} on the particle (B), which is measured by an observer at the point of the particle (A), is represented as $F_{\mathrm{inert}} = ml\ddot{a}/a = ml\left(\dot{H} + H^2\right)$ [45, 47]. In the case of a Big Rip singularity with H in Eq. (III.1), $F_{\mathrm{inert}} \to \infty$ when $t \to t_{\mathrm{s}}$. Moreover. for Little Rip scenario with H described in Table III, $F_{\mathrm{inert}} \to \infty$ when $t \to \infty$. Furthermore, in Pseudo-Rip scenario with H presented in Table III, $F_{\mathrm{inert}} \to F_{\mathrm{inert},\infty}^{\mathrm{PR}} \equiv mlH_{\mathrm{PL}}^2 < \infty$ when $t \to \infty$. Thus, F_{inert} approaches a finite value asymptotically. This is because $H \to H_{\mathrm{PR}}$ and $\dot{H} \to 0$.

If a force F_{b} to bound two particles is smaller than a positive inertial force $F_{\mathrm{inert}}(> 0)$, the two particle bound system is disintegrated. As an example, we examine the Earth-Sun (ES) system. When $F_{\mathrm{inert},\infty}^{\mathrm{PR}} > F_{\mathrm{b}}^{\mathrm{ES}} = GM_{\oplus}M_{\odot}/r_{\oplus-\odot}^2 = 4.37 \times 10^{16}\,\mathrm{GeV}^2$, which is the bound force in the ES system, the ES system is dissociated, so that Pseudo-Rip scenario can be satisfied. Here, we have used the mass of Earth $M_{\oplus} = 3.357 \times 10^{51}\,\mathrm{GeV}$ [73] and that of Sun $M_{\odot} = 1.116 \times 10^{57}\,\mathrm{GeV}$ [73], and we have set $m = M_{\oplus}$ and $l = r_{\oplus-\odot} = 1\mathrm{AU} = 7.5812 \times 10^{26}\,\mathrm{GeV}^{-1}$ [73] (the Astronomical unit, namely, the distance between Earth and Sun). In this case, we acquire $H_{\mathrm{PR}} > \sqrt{GM_{\odot}/r_{\oplus-\odot}^3} = 1.31 \times 10^{-31}\,\mathrm{GeV}$. This is consistent with the observations on the present value of w_{DE} in Pseudo-Rip scenario because this is much stronger than the constraint $H_{\mathrm{PR}} \geq 2.96 \times 10^{-42}\,\mathrm{GeV}$ given above, which originates from the observations on the value of w_{DE} at the present time.

It is also remarked that in the process of collapse of the star, the time-dependent matter instability can happen not only for $F(R)$ gravity [75, 76] but also $F(T)$ gravity.

IV. HIGHER-DIMENSIONAL THEORIES

In this section, we construct four-dimensional effective $F(T)$ gravity theories from the five-dimensional Kaluza-Klein (KK) [56–58] and Randall-Sundrum (RS) [59, 60] theories by following the investigations in Ref. [27].

A. Five-dimensional KK theory

First, we derive the effective $F(T)$ gravity theories in the four-dimensional space-time from the KK theory in the five-dimensional space-time. It is supposed that in $F(T)$ gravity, the ordinary procedure of the KK reduction [56–58] can be executed from the five-dimensional space-time to the four-dimensional space-time. In this process, one dimension of space is compacted into a small circle, while the four-dimensional space-time is infinitely extended. The radius of the fifth dimension is set to be around the Planck length so that the KK effects cannot appear. Consequently, the size of the circle is small enough for the phenomena in the quite low energy scale not to be seen. From now on, we concentrate on the gravity sector in the action, and therefore the matter sector is neglected.

The five-dimensional action in $F(T)$ gravity is [77]

$$^{(5)}S = \int d^5x \left|^{(5)}e\right| \frac{F(^{(5)}T)}{2\kappa_5^2}, \tag{IV.1}$$

$$^{(5)}T \equiv \frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{cba} - T_{ab}{}^{a}T^{cb}{}_{c}. \tag{IV.2}$$

Here, $^{(5)}T$ is the torsion scalar, the Latin indices are $(a, b, \cdots = 0, 1, 2, 3, 5)$ with "5" the fifth-coordinate component, $^{(5)}e = \sqrt{^{(5)}g}$ with $^{(5)}g$ the determinant of $^{(5)}g_{\mu\nu}$, and $\kappa_5^2 \equiv 8\pi G_5 = \left(^{(5)}M_{\mathrm{Pl}}\right)^{-3}$ with G_5 the gravitational constant and $M_{\mathrm{Pl}}^{(5)}$ the Planck mass. The superscript "(5)" or the subscription "5" shows the quantities in the five-dimensional space-time. The representation of $^{(5)}T$ is equivalent to that of the torsion scalar in the four-dimensional space-time. The five-dimensional metric is given by $^{(5)}g_{\mu\nu} = \mathrm{diag}(g_{\mu\nu}, -\psi^2)$, where ψ is a dimensionless and (spatially) homogeneous scalar field (namely, it only has the time dependence).

The four-dimensional effective action becomes

$$S_{\mathrm{KK}}^{(\mathrm{eff})} = \int d^4x |e| \frac{1}{2\kappa^2} \psi F(T + \psi^{-2}\partial_\mu\psi\partial^\mu\psi) \,, \tag{IV.3}$$

where we have used $e_a^A = \mathrm{diag}(1,1,1,1,\psi)$ and $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1,-1)$. In the case that $F(T) = T - 2\Lambda_4$ with $\Lambda_4(> 0)$ the positive cosmological constant in the four-dimensional space-time, by defining another scalar field ξ as $\psi \equiv (1/4)\,\xi^2$, we find that the action in Eq. (IV.3) is described as [56]

$$S_{\mathrm{KK}}^{(\mathrm{eff})}|_{F(T)=T-2\Lambda_4} = \int d^4x |e| \frac{1}{\kappa^2} \left[\frac{1}{8}\xi^2 T + \frac{1}{2}\partial_\mu\xi\partial^\mu\xi - \Lambda_4 \right] \,. \tag{IV.4}$$

In the flat FLRW background, from the action in Eq. (IV.4), the gravitational field equations are given by $(1/2)\dot{\xi}^2 - (3/4)\,H^2\xi^2 + \Lambda_4 = 0$ and $\dot{\xi}^2 + H\xi\dot{\xi} + (1/2)\,\dot{H}\xi^2 = 0$ [78], and the equation of motion in terms of ξ is written as $\ddot{\xi} + 3H\dot{\xi} + (3/2)\,H^2\xi = 0$. By using the gravitational field equations, we have $(3/2)\,H^2\xi^2 - 2\Lambda_4 + H\xi\dot{\xi} + (1/2)\,\dot{H}\xi^2 = 0$. Its solution is $H = H_{\mathrm{inf}} = \mathrm{constant}(> 0)$, where H_{inf} is the Hubble parameter at the inflationary stage, and $\xi = \xi_1\,(t/\bar{t}) + \xi_2$ with ξ_1 and $\xi_2(> 0)$ constants, where $\bar{t}$ denotes a time. Thus, when $t \to 0$, inflation with the de Sitter expansion can be realized, where $H_{\mathrm{inf}} \approx (2/\xi_2)\sqrt{\Lambda_4/3}$, $a \approx \bar{a}\exp(H_{\mathrm{inf}}t)$, and $\xi \approx \xi_2$. Moreover, with the equation of motion in terms of ξ, we acquire $\xi_1 \approx -(1/2)\,\xi_2 H_{\mathrm{inf}}\bar{t} \approx -\sqrt{\Lambda_4/3}\bar{t}$.

B. RS brane-world model

Next, we deduce the effective $F(T)$ gravity theories in the four-dimensional space-time from the RS brane-world model in the five-dimensional space-time. There exist two branes in the RS type-I model [59]: A positive tension brane located at $y = 0$ and a negative one located at $y = u$, where y means the fifth dimension.

The five-dimensional metric is expressed as

$$ds^2 = \exp\left(-2\frac{|y|}{l}\right) g_{\mu\nu}(x)dx^\mu dx^\nu + dy^2 \,, \tag{IV.5}$$

where $l = \sqrt{-6/\Lambda_5}$, $\exp(-2|y|/l)$ is the warp factor, and $\Lambda_5(< 0)$ is the negative cosmological constant in the bulk. When $u \to \infty$, the RS type-I model is reduced to the RS type-II model [60]. In this model, there is only one positive tension brane in the anti-de Sitter bulk space. In Ref. [79], the gravitational field equation on the brane has been presented for the RS type-II model. It corresponds to the induced equation, i.e., the Gauss-Codazzi equation, on the brane, and it is derived by using the Israel's junction conditions on the brane and the Z_2 symmetry of $y \leftrightarrow -y$. This procedure has recently been considered in teleparallelism [80]. The vector part of the torsion tensor in the bulk is projected on the brane, so that new terms, which do not exist in the curvature gravity, can emerge.

In the flat FLRW background, the Friedmann equation on the brane is given by

$$H^2 \frac{dF(T)}{dT} = -\frac{1}{12}\left[F(T) - 4\Lambda - 2\kappa^2\rho_{\mathrm{M}} - \left(\frac{\kappa_5^2}{2}\right)^2 \mathcal{I}\rho_{\mathrm{M}}^2 \right] \,. \tag{IV.6}$$

Here, $\mathcal{I} \equiv \left(11 - 60w_{\mathrm{M}} + 93w_{\mathrm{M}}^2\right)/4$ with $w_{\mathrm{M}} \equiv P_{\mathrm{M}}/\rho_{\mathrm{M}}$ the EoS for matter, where ρ_{M} and P_{M} are the energy density and pressure of matter, respectively. In the expression of $\mathcal{I}$, the novel contributions in teleparallelism are included (there are not these terms in the curvature gravity). The effective cosmological constant on the brane reads $\Lambda \equiv \Lambda_5 + \left(\kappa_5^2/2\right)^2\lambda^2$, where $\lambda(> 0)$ is the brane tension and we obtain the relation $G = [1/(3\pi)]\left(\kappa_5^2/2\right)^2\lambda$.

In the following, we consider the situation that the dark energy is dominant and hence the contribution of matter is negligible. If $F(T) = T - 2\Lambda_5$, with $T = -6H^2$, we get a solution of the de Sitter expansion as $H = H_{\mathrm{DE}} = \sqrt{\Lambda_5 + \kappa_5^4\lambda^2/6}$ and $a(t) = a_{\mathrm{DE}}\exp(H_{\mathrm{DE}}t)$, where H_{DE} and $a_{\mathrm{DE}}(> 0)$ are constants. Accordingly, the late-time cosmic acceleration can happen. In addition, when $F(T) = \left(T^2/\bar{M}^2\right) + \eta\Lambda_5$ with $\bar{M}$ a mass scale and η a constant, we find a de Sitter solution with the constant Hubble parameter

$$H = H_{\mathrm{DE}} = \left\{ \frac{\bar{M}^2}{108}\left[(\eta - 4)\Lambda_5 - 4\left(\frac{\kappa_5^2}{2}\right)^2\lambda^2 \right] \right\}^{1/4} \,. \tag{IV.7}$$

In this expression, the content of the 4th root has to be positive. Therefore, we obtain a constraint on η, given by $\eta \geq 4 + \left(\kappa_5^2\lambda^2\right)/\Lambda_5$.

V. SUMMARY

In the present paper, we have stated various cosmological issues as well as theoretical properties in $F(T)$ gravity. First, we have investigated the cosmological evolution of the EoS for dark energy w_{DE}. We have constructed an $F(T)$ gravity model consisting of an exponential term and a logarithmic one, in which the crossing of the phantom divide can occur.

Next, we have found that the Type I and IV finite-time future singularities can appear, and reconstructed an $F(T)$ gravity model in which there exist the finite-time future singularities. We have also demonstrated that by adding a power-law term of T^β ($\beta > 1$) like T^2, the finite-time future singularities can be cured, similarly to that in $F(R)$ gravity. Furthermore, we have explored $F(T)$ gravity models in which the following cosmological scenario is satisfied: power-law inflation, the ΛCDM model, the Little Rip scenario, and the Pseudo Rip scenario.

Moreover, we have analyzed four-dimensional effective action of $F(T)$ gravity originating from the five-dimensional KK theories and RS models. We have derived the four-dimensional effective action with a coupling of the torsion scalar to a scalar field through the KK reduction to the four-dimensional space-time from the five-dimensional one. We have shown that in this theory, inflation can occur. We have also found that in the RS type-II model with the four-dimensional FLRW brane, $F(T)$ gravity influences on the four-dimensional FLRW brane. We have seen that for this model, the late-time cosmic acceleration can be realized. Here, inflation or the late-time cosmic acceleration can happen thanks to the torsion effect, and not by the curvature one, so that these KK theories and RS models can be considered to be constructed by not the scalar curvature but the torsion one in teleparallel gravity.

It should be cautioned that there is no local Lorentz invariance in $F(T)$ gravity as indicated in Refs. [81, 82], and this theory is acausal [83–85]. These are the most crucial points for $F(T)$ gravity theory. Thus, these problems have to further be considered seriously.

Finally, we mention a number of other cosmological subjects have been studied in $F(T)$ gravity. As examples, the authors works are raised: Trace-anomaly driven inflation [86], gravitational wave modes [87, 88], conformal symmetry [89], thermodynamics [90, 91], and the generation of the large-scale magnetic fields [92, 93][2]. It is expected that through such various cosmological investigations in $F(T)$ gravity, the clues to find novel viability conditions for $F(T)$ gravity as an alternative gravity theory to general relativity can be acquired.

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[2] Also in $F(R)$ gravity, trace-anomaly driven inflation [94, 95], thermodynamics [31, 96–99], and the generation of the large-scale magnetic fields [100, 101] have been studied.

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